FOLIATIONS AND POLYNOMIAL DIFFEOMORPHISMS OF $\mathbb{R}^3$

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ABSTRACT. Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map and let $\text{Spec}(Y)$ denote the set of eigenvalues of the derivative DY_p, when p varies in $\mathbb{R}^3$. We begin proving that if, for some $\epsilon > 0$, $\text{Spec}(Y) \cap (-\epsilon, \epsilon) = \emptyset$, then the foliation $\mathcal{F}(k)$, with $k \in \{f, g, h\}$, made up by the level surfaces $\{k = \text{constant}\}$, consists just of planes. As a consequence, we prove a bijectivity result related to the three-dimensional case of Jelonek's Jacobian Conjecture for polynomial maps of $\mathbb{R}^n$.

1. Introduction

Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map and let $\text{Spec}(Y)$ be the set of (complex) eigenvalues of the derivative DY_p when p varies in $\mathbb{R}^3$. If for all $p \in \mathbb{R}^3$, DY_p is non singular, (that is, $0 \notin \text{Spec}(Y)$) then it follows from the inverse function theorem that:

for each $k \in \{f, g, h\}$, the level surfaces $\{k = \text{constant}\}$ make up a codimension one C^2-foliation $\mathcal{F}(k)$ on $\mathbb{R}^3$. Our first result is the following

Theorem 1.1. *If, for some $\epsilon > 0$, $\text{Spec}(Y) \cap (-\epsilon, \epsilon) = \emptyset$, then $\mathcal{F}(k)$, $k \in \{f, g, h\}$, is a foliation by planes. Consequently, there is a foliation F_k in $\mathbb{R}^2$ such that $\mathcal{F}(k)$ is conjugate to the product of F_k by $\mathbb{R}$.*

To state our next results, we need to introduce some concepts. Let $Y : M \to N$ be a continuous map of locally compact spaces. We say that the mapping Y is *not proper at a point* $y \in N$, if there is no neighborhood U of the point y such that the set $Y^{-1}(\overline{U}))$ is compact.

The set S_Y of points at which the map Y is not proper indicates how the map Y differs from a proper map. In particular Y is proper if and only if this set is empty. Moreover, if $Y(M)$ is open, then S_Y contains the border of the set $Y(M)$. The set S_Y is the minimal set S with a property that the mapping $Y : M \setminus Y^{-1}(S) \to N \setminus S$ is proper.

Jelonek proved in [20] that: if $Y : \mathbb{R}^n \to \mathbb{R}^n$ is a real polynomial mapping with nonzero Jacobian everywhere and $\text{codim}(S_Y) \geq 3$, then Y is a bijection (and consequently $S_Y = \emptyset$).

On the other hand, the example of Pinchuk (see [26]) shows that there are real polynomial mappings, which are not injective, with nonzero Jacobian

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everywhere and with $\operatorname{codim}(S_Y) = 1$. Hence the only interesting case is that of $\operatorname{codim}(S_Y) = 2$ and we can state:

Jelonek's Real Jacobian Conjecture. *Let $Y : \mathbb{R}^n \to \mathbb{R}^n$ be a real polynomial mapping with nonzero Jacobian everywhere. If $\operatorname{codim}(S_Y) \geq 2$ then Y is a bijection (and consequently $S_Y = \emptyset$).*

Jelonek [20] proved that his conjecture is true in dimension two. Consequently, the first interesting case is $n = 3$ and $\dim(S_Y) = 1$.

Jelonek's Real Jacobian Conjecture is closely connected with the following famous Keller Jacobian Conjecture:

Jacobian Conjecture. *Let $Y : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with nonzero Jacobian everywhere, then Y is an isomorphism.*

More precisely, Jelonek proved in [20] that his Real Jacobian Conjecture in dimension $2n$ implies the Jacobian Conjecture in (complex) dimension n. The corresponding Jelonek's arguments and some well known results ([2], [8], [10], [29]) will be used to obtain in section 3 the following version of the Reduction Theorem

Theorem 1.2. *Let $X_i : \mathbb{C}^n \to \mathbb{C}$ denote the canonical i-coordinate function. If F, with $\operatorname{codim}(S_F) \geq 2$, is injective for all $n \geq 2$ and all polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ of the form*

$$F = (-X_1 + H_1, -X_2 + H_2, \ldots, -X_n + H_n)$$

where each $H_i : \mathbb{R}^n \to \mathbb{R}$ is either zero or homogeneous of degree 3, and the Jacobian matrix JH (with $H = (H_1, H_2, \ldots, H_n)$) is nilpotent, then the Jacobian Conjecture is true.

Notice that in theorem above $\operatorname{Spec}(F) = \{-1\}$.

Related with Theorem 1.2 and Jelonek conjecture we prove the following.

Theorem 1.3. *Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map such that $\operatorname{Spec}(Y) \cap [0, \varepsilon) = \emptyset$, for some $\varepsilon > 0$. If $\operatorname{codim}(S_Y) \geq 2$ then Y is a bijection.*

This result partially extends also the bi-dimensional results of [7] and [11] (see also [5] – [6], [12], [15] – [23], [25]).

2. Half-Reeb Components and The Spectral Condition

Let us recall the definition of a vanishing cycle stated in conformity with our needs. Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map such that, for all $p \in \mathbb{R}^3$, DY_p is non-singular. Given $k \in \{f, g, h\}$, a *vanishing cycle* for the foliation $\mathcal{F}(k)$ is a C^2-embedding $f_0 : S^1 \to \mathbb{R}^3$ such that:

(a) $f_0(S^1)$ is contained in a leaf L_0 but it is not homotopic to a point in L_0;

(b) f_0 can be extended to a C^2-embedding $f : [0,1] \times S^1 \to \mathbb{R}^3$, $f(t, x) = f_t(x)$, such that for all $t > 0$, there is a 2-disc D_t is contained in a leaf L_t, such that $\partial D_t = f_t(S^1)$;

(c) for all $x \in S^1$, the curve $t \mapsto f(t, x)$ is transversal to the foliation $\mathcal{F}(k)$ and, for all $t \in (0, 1)$, D_t depends continuously on t.

We say that the leaf L_0 *supports* the vanishing cycle f_0 and that f is the map associated to f_0.

The *half-Reeb component for* $\mathcal{F}(k)$ (or simply the *hRc* for $\mathcal{F}(k)$) associated to the vanishing cycle f_0 is the region

$$\mathcal{A} = \left(\bigcup_{t\in(0,1]} D_t \right) \cup L \cup f_0(S^1)$$

where L is the connected component of $L_0 - f_0(S^1)$ contained in the closure of $\cup_{t\in(0,1]} D_t$. The transversal section $A = f([0,1]\times S^1)$ to the foliation $\mathcal{F}(k)$ is called the *compact face* of $\mathcal{A}$ and the leaf $L \cup f_0(S^1)$ of $\mathcal{F}(k)|_{\mathcal{A}}$ is called the *non-compact face* of $\mathcal{A}$.

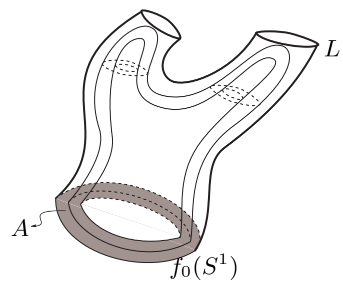


FIGURE 1. A half-Reeb component.

Remark 2.1.

(1) It will be seen in Proposition 2.2, that if $\mathcal{F}(k)$, $k \in \{f, g, h\}$, has a leaf which is not homeomorphic to the plane, then $\mathcal{F}(k)$ has a half-Reeb component.
(2) The connection between half-Reeb components and the spectral condition on Y (that is, $\mathrm{Spec}(Y) \cap (-\varepsilon, \varepsilon) = \emptyset$) is given by Theorem 1.1.

The following proposition is obtained by using classical arguments of Foliation Theory (see [4] and [13]). For sake of completeness we give the main lines of its proof. Let $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ denote the closed 2-disc.

Proposition 2.2. *If* $\mathcal{F}(k)$, *with* $k \in \{f, g, h\}$, *has a leaf* L *which is not homeomorphic to the plane, then* $\mathcal{F}(k)$ *has a vanishing cycle.*

Proof. Let $\eta : S^1 \to L$ be an embedding which is not null homotopic in L. Since η is null homotopic in $\mathbb{R}^3$, we may extend it to a C^2-immersion $\eta : D^2 \to \mathbb{R}^3$, which is in general position with respect to $\mathcal{F}(k)$. It this way we are supposing that the contact set C_η, made up by the points of D^2 at which η meets tangentially $\mathcal{F}(k)$, is finite and is contained in $D^2 \setminus S^1$.

Via η, the foliation $\mathcal{F}(k)$ induces a foliation $\mathcal{G}$ (with singularities) on D^2. We claim that it is possible construct a vector field G on D^2 such that the foliation $\mathcal{G}$ is induced by G. In fact, as η is in general position with respect to $\mathcal{F}(k)$, the foliation $\mathcal{G}$ has finitely many singularities each of which is locally

topologically equivalent either to a center or to a saddle point of a vector field. This implies that $\mathcal{G}$ is locally orientable everywhere. As D^2 is simple connected, $\mathcal{G}$ is globally orientable. This proves the existence of the vector field G. Certainly, we may assume that η has been chosen so that no pair of singularities of G is taken by η into the same leaf of $\mathcal{F}(k)$; in other words, G has no saddle connections.

We claim that G has no limit cycles. In fact, otherwise, the Poincaré-Bendixon theorem would imply that there is a orbit of G which spirals towards a limit cycle C. Hence, the leaf of $\mathcal{F}(k)$ containing C would have a non trivial holonomy group. This contradiction proves our claim.

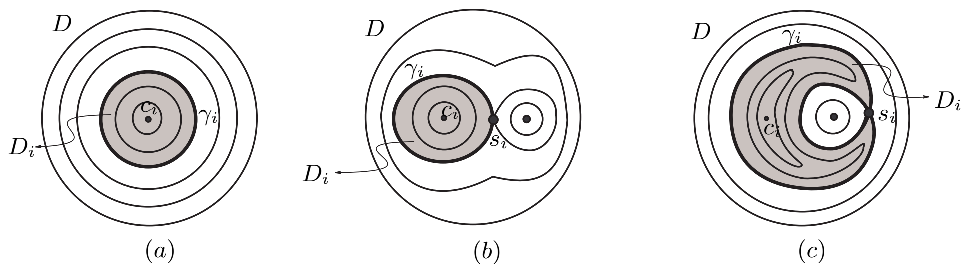


FIGURE 2.

Let $c_1, \ldots, c_\ell$ be the center singularities of G. Given $i \in \{1, \ldots, \ell\}$, there exists a G-invariant open 2-disc $D_i \subset D^2$ such that:

(a1) $c_i \in D_i$ and every orbit of G passing through a point in $D_i \setminus \{c_i\}$ is a closed orbit;
(a2) for every closed orbit $\gamma \subset D_i$ of G, $\eta(\gamma)$ is homotopic to a point in its corresponding leaf of $\mathcal{F}(k)$.
(a3) the 2-disc Di is the biggest one satisfying properties (a1) and (a2) above.

Notice that the frontier γ_i of D_i has to be G-invariant. We claim that

(b) If, for some $i \in \{1, 2, \ldots, \ell\}$, γ_i is a closed orbit of G, then $\eta(\gamma_i)$ is a vanishing cycle (see Fig. 2(a)), and the proposition is proved.

In fact, if γ_i is a closed orbit of G such that $\eta(\gamma_i)$ is homotopic to a point in its corresponding leaf, then, by a well known result of foliation theory, there exists a neighborhood $V_i \subset D^2$ of γ_i such that the image by η of every orbit of G, contained in V_i, is homotopic to a point in its corresponding leaf. This contradiction with the maximality of D_i proves (b).

Therefore, we may suppose, from now on, that:

(c) for every $i \in \{1, \ldots, \ell\}$, γ_i is either the union of a saddle singularity s_i of G and one of its separatrices or the union of a saddle singularity s_i and its two separatrices, see (b) and (c) of Figure 2.

By studying the phase portrait of G, we may conclude that

(d) if (b) is not satisfied, there must exist $i \in \{1, 2, \ldots, \ell\}$ such that γ_i is the union of a saddle singularity s_i of G and one of its separatrices.

We claim that:

(e.1) If $\eta(\gamma_i)$ is homotopic to a point in its corresponding leaf, then η can be deformed to a C^2-immersion $\tilde{\eta} : D^2 \to \mathbb{R}^3$ which is in general position with respect to $\mathcal{F}(k)$ and such that $\#C_{\tilde{\eta}} < \#C_\eta$;

(e.2) If $\eta(\gamma_i)$ is a not homotopic to a point in its corresponding leaf, then η can be deformed to a C^2-immersion $\tilde{\eta} : D^2 \to \mathbb{R}^3$ for which (b) above is satisfied.

In fact, let us prove (e.1). By using Rosenberg's arguments (see [27, pag. 137]), via a deformation of η, supported in a neighborhood of $\overline{D}_i$, we can eliminate the saddle singularity s_i and the center singularity c_i. the proof of (e.2) is similar and will be omitted.

Using (e.1) as many times as necessary, it follows from (d) that we will arrive to the situation considered in (e.2). this proves the proposition. □

Lemma 2.3. *Let $\mathcal{F}_i$, $i = 1, 2, 3$, be a C^2 foliation on $\mathbb{R}^3$ without holonomy such that for $j \neq i$, $\mathcal{F}_j$ is transversal to $\mathcal{F}_i$. Let L be a leaf of $\mathcal{F}_1$. If $\mathcal{F}_2|_L$ is the foliation on L that is induced by $\mathcal{F}_2$, then every leaf of $\mathcal{F}_2|_L$ is homeomorphic to $\mathbb{R}$.*

Proof. Suppose that there exists a leaf S of $\mathcal{F}_2|_L$, homeomorphic to S^1. The fact that $\mathcal{F}_2$ is without holonomy and $\mathcal{F}_3|_L$ is transversal to $\mathcal{F}_2|_L$ implies that there exists a neighborhood C of S in L such that every leaf of $\mathcal{F}_2|_L$ passing through a point in C is homeomorphic to S^1 and is not homotopic to a point in L. Moreover, the leaves of $\mathcal{F}_3|_L$ restricted to C are curves starting at one connected components of ∂C, and ending at the other one.

Let D be a smoothly immersed open 2-disc containing S, which we may assume to be in general position with respect to $\mathcal{F}_3$. Let $\mathcal{G}_3$ be the foliation (with singularities) of D which is induced by $\mathcal{F}_3$. Then, $\mathcal{G}_3$ is transversal to S.

We claim that $\mathcal{G}_3$ has no limit cycles, otherwise, the Poincaré-Bendixon theorem implies that there is a leaf of $\mathcal{G}_3$ which spirals towards a limit cycle γ. Hence, the leaf of $\mathcal{F}_3$ containing γ would have a non trivial holonomy group. This contradiction proves our claim. It follows from the claim above that $\mathcal{G}_3$, has exactly one singularity. Since $\mathcal{G}_3$ is transversal to S, this singularity is an attractor. But D in general position with respect to $\mathcal{F}_3$ means that $\mathcal{G}_3$ has a finite number of singularities, each of which is either a center or a saddle point. This contradiction concludes the proof. □

Remark 2.4. Let $k \in \{f, g, h\}$. As k is a submersion, the foliation $\mathcal{F}(k)$ is without holonomy.

Corollary 2.5. *Let $\{i, j, k\}$ be an arbitrary permutation of $\{f, g, h\}$. If L is a leaf of $\mathcal{F}(i)$ and l is a leaf of $\mathcal{F}(j)|_L$ then $k|_l$ is regular; in this way $\mathcal{F}(j)|_L$ and $\mathcal{F}(k)|_L$ are transversal to each other.*

For each $\theta \in \mathbb{R}$ let $T_\theta,\ S_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations defined by the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},$$

respectively. Note that T_θ (resp. S_θ) restricted to the xy-plane (resp. xz-plane) is the rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Let $\Pi : \mathbb{R}^3 \to \mathbb{R}$ given by $\Pi(x, y, z) = x$. The following proposition will be needed.

Proposition 2.6. *Let $Y = (f, g, h)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map such that $0 \notin \mathrm{Spec}(Y)$ and $\mathcal{A}$ be a hRc of $\mathcal{F}(f)$. If $\Pi(\mathcal{A})$ is bounded, then there is an $\epsilon > 0$ and $K_\theta \in \{S_\theta, T_\theta\}$ such that, for all $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$, $\mathcal{F}(f_\theta)$ has a hRc $\mathcal{A}_\theta$ such that $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length, where $(f_\theta, g_\theta, h_\theta) = K_\theta \circ Y \circ K_{-\theta}$.*

Proof. If $\Pi(\mathcal{A})$ is bounded, then either $\{y : (x, y, z) \in \mathcal{A}\}$ or $\{z : (x, y, z) \in \mathcal{A}\}$ is an interval of infinite length. We are going to show that, if $\{y : (x, y, z) \in \mathcal{A}\}$ is an interval of infinite length, then, for $K_\theta = T_\theta$, $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length. The proof of the other case is analogous in which case, the proposition is satisfied for $K_\theta = S_\theta$. Then, assume that $\{y : (x, y, z) \in \mathcal{A}\}$ is an interval of infinite length.

(a) Let $\theta \in \mathbb{R}$ be such that, for all $m \in \mathbb{Z}$, $\theta \neq \frac{m\pi}{2}$. Then $\mathcal{F}(f_\theta)$ is transversal to both $T_\theta(\mathcal{F}(f))$ and $T_\theta(\mathcal{F}(g))$.

In fact, assume by contradiction that there exist $p \in \mathbb{R}^3$ such that $L_{T_\theta(p)}(f_\theta)$ and $T_\theta(L_p(f))$ (the leaves through $T_\theta(p)$ of $\mathcal{F}(f_\theta)$ and $T_\theta(\mathcal{F}(f))$, respectively) are tangent at $T_\theta(p)$. This implies that every C^1 curve in $T_\theta(L_p(f))$ passing through $T_\theta(p)$ is tangent to $L_{T_\theta(p)}(f_\theta)$ at $T_\theta(p)$. But, we will exhibit a C^1 curve $\alpha_\theta : (-1, 1) \to T_\theta(L_p(f))$ passing through $T_\theta(p)$ which is not tangent to $L_{T_\theta(p)}(f_\theta)$ at $T_\theta(p)$. Indeed, we consider $\alpha_\theta : (-1, 1) \to T_\theta(L_p(f))$ defined by $\alpha_\theta = T_\theta \circ \alpha$ where $\alpha : (-1, 1) \to \mathbb{R}^3$ is a C^1 curve contained in $L_p(f) \cap L_p(h)$ with $\alpha(0) = p$ and $\alpha'(0) \neq 0$. By Corollary 2.5, $(g \circ \alpha)'(0) \neq 0$. Hence, as $f(\alpha(t)) \equiv$ constant, $\sin\theta \neq 0$ and

$$(f_\theta \circ \alpha_\theta)(t) = (\cos\theta) f(\alpha(t)) - (\sin\theta) g(\alpha(t)), \quad t \in (-1, 1),$$

we obtain that

$$(f_\theta \circ \alpha_\theta)'(0) = -\sin\theta\, (g \circ \alpha)'(0) \neq 0$$

and so α_θ is not tangent to $L_{T_\theta(p)}(f_\theta)$ at $T_\theta(p) = \alpha_\theta(0)$. This contradiction proves that $\mathcal{F}(f_\theta)$ is transversal to $T_\theta(\mathcal{F}(f))$. Similarly we prove that $\mathcal{F}(f_\theta)$ is transversal to $T_\theta(\mathcal{F}(g))$.

Take Σ diffeomorphic to the open annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$, transversal to $\mathcal{F}(f)$ and containing the compact face A of $\mathcal{A}$. Since, for θ enough

small, Y_θ and T_θ are C^1 close to Y and to the identity T_O, respectively, we can take Σ so that

(b) there exist $\varepsilon > 0$ such that, for all $\theta \in (-\varepsilon, \varepsilon)$, $T_\theta(\Sigma)$ is transversal to both $T_\theta(\mathcal{F}(f))$ and $\mathcal{F}(f_\theta)$.

Let $\mathcal{G}_\theta$ be the foliation in $T_\theta(\Sigma)$ which is induced by $\mathcal{F}(f_\theta)$. As $\mathcal{F}(f_\theta)$ is without holonomy, we can take $\varepsilon > 0$ so that

(c) for all $\theta \in (-\varepsilon, \varepsilon)$, there exist open cylinders A_θ^-, $A_\theta^+ \subset T_\theta(A_0)$ made up by closed trajectories of $\mathcal{G}_\theta$ such that $A_\theta^- \subset T_\theta(A)$, $A_\theta^+ \cap T_\theta(A) = \emptyset$, $A_\theta^- \cap T_\theta(\partial A) = \emptyset = A_\theta^+ \cap T_\theta(\partial A)$ and both A_θ^- and A_θ^+ are the biggest cylinders with these properties.

We claim that:

(d) every leaf of $\mathcal{G}_\theta$ contained in A_θ^+ is not homotopic to a point in its corresponding leaf of $\mathcal{F}(f_\theta)$.

In fact, assume by contradiction that there exist a leaf γ of $\mathcal{G}_\theta$ contained in A_θ^+ and bounding a closed 2-disc $D(\gamma)$ contained in a leaf of $\mathcal{F}(f_\theta)$. If L is the non-compact face of $\mathcal{A}$ and $\tilde{D}(\gamma) \subset T_\theta(A_0)$ is the disc bounded by γ, then the 2–sphere $D(\gamma) \cup (\tilde{D}(\gamma))$ meets $T_\theta(L)$ at a circle contained in $\tilde{D}(\gamma)$. Therefore, as the referred 2–sphere separates $\mathbb{R}^3$, $T_\theta(L)$ has to meet $D(\gamma)$ and so there exists a closed 2-disc $D_0(\gamma) \subset D(\gamma)$ such that $\partial D_0(\gamma) = D(\gamma) \cap T_\theta(L)$. Consequently, there is at least one point in $D_0(\gamma)$ where $T_\theta(\mathcal{F}(f))$ and $\mathcal{F}(f_\theta)$ are tangent, contradicting (a). This proves (d).

By using a similar argument we may also obtain that

(e) every leaf of $\mathcal{G}_\theta$ contained in A_θ^- is homotopic to a point in its corresponding leaf of $\mathcal{F}(f_\theta)$.

In what follows of this proof, every time that we refer to Lemma 2.3, we will be assuming that it is been applied to the three foliations $\mathcal{F}(f_\theta)$, $T_\theta(\mathcal{F}(f))$ and $T_\theta(\mathcal{F}(g))$.

From (c), (e) and Lemma 2.3, we obtain that there exists a leaf γ of $\mathcal{G}_\theta$ contained in $T_\theta(A_0) \setminus (A_\theta^- \cup A_\theta^+)$ which is a vanishing cycle of $\mathcal{F}(f_\theta)$ and such that

(f) $\gamma \cap T_\theta(\partial A)$ is a nonempty finite set.

Let $\mathcal{A}_\theta$ be the *hRc* of $\mathcal{F}(f_\theta)$ with non-compact face L_θ and compact face contained in $T_\theta(\Sigma)$ and bounded by γ. Notice that $L = L_O$. Let $a_1, \ldots, a_{2\ell} \in A \cap L$ be such that

$$\gamma \cap T_\theta(A \cap L) = \{T_\theta(a_1), \ldots, T_\theta(a_{2\ell})\}$$

Up to small deformation of Σ, if necessary, we may assume that, for all $i = 1, \ldots, 2\ell$, the connected component Γ_i of $T_\theta(L) \cap L_\theta$ that contains $T_\theta(a_i)$ is a regular curve (not reduced to a single point).

We claim that

(g) There exists $i_0 \in \{1, \ldots, 2\ell\}$ such that Γ_{i_0} is non-compact.

In fact, suppose by contradiction that Γ_i is compact for every $i \in \{1, \ldots, 2\ell\}$. Recall that L_θ (resp. $T_\theta(L)$) is the noncompact face of $\mathcal{A}_\theta$ (resp. of $T_\theta(\mathcal{A})$).

Let $U(L_\theta)$ (resp. $U(T_\theta(L))$ be the unbounded connected component of $L_\theta \setminus (\cup_{i=1}^{2\ell}\Gamma_i)$ (resp. of $T_\theta(L) \setminus (\cup_{i=1}^{2\ell}\Gamma_i)$). As $U(L_\theta) \cap U(T_\theta(L)) = \emptyset$ and both $\partial(\mathcal{A}_\theta)$ and $\partial(T_\theta(\mathcal{A}))$ separate $\mathbb{R}^3$, we have that either

$$U(L_\theta) \subset T_\theta(\mathcal{A}) \text{ or } U(T_\theta(L)) \subset \mathcal{A}_\theta\,,$$

respectively. If $U(L_\theta) \subset T_\theta(\mathcal{A})$ then, since all leaves of $T_\theta(\mathcal{F}(f))|_{T_\theta(\mathcal{A})}$ passing through points in the interior of $T_\theta(\mathcal{A})$ are closed 2-discs, it follows that $T_\theta(\mathcal{F}(f))|_{L_\theta}$ has infinitely many leaves which are homeomorphic to S^1, contradicting Lemma 2.3. Analogously, if $U(T_\theta(L)) \subset \mathcal{A}_\theta$ we obtain a contradiction with Lemma 2.3. This proves (g).

Now we claim that

(h) $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length.

Let $\Gamma_i = \Gamma_{i_0}$ be as in (g). As $\Pi(\Gamma_i) \subset \Pi(\mathcal{A}_\theta) \cap \Pi(T_\theta(\mathcal{A}))$, it is enough to prove that $\Pi(\Gamma_i)$ is an interval of infinite length. Since $\Gamma_i \subset L_\theta \cap T_\theta(L)$, we have that $T_\theta^{-1}(\Gamma_i) \subset L \subset \mathcal{A}$, consequently $\Pi(T_\theta^{-1}(\Gamma_i)) \subset \Pi(\mathcal{A})$. Now, if $\Pi(\Gamma_i)$ was bounded, then the subinterval $\Pi(T_\theta^{-1}(\Gamma_i))$ of $\Pi(\mathcal{A})$ would have infinite length, contradicting the assumption that $\Pi(\mathcal{A})$ is bounded. This proves (h) and concludes the proof of this proposition. $\square$

Proof of Theorem 1.1. By Palmeira's theorem, see [24], it is sufficient to show that $\mathcal{F}(k),\ k \in \{f, g, h\}$, is a foliation by planes. Suppose by contradiction that $\mathcal{F}(f)$ has a leaf which is not homeomorphic to $\mathbb{R}^2$. It follows, from Proposition 2.2, that $\mathcal{F}(f)$ has a half-Reeb component $\mathcal{A}$. Hereafter we will use the fact that existence of a half-Reeb component and the assumptions of Theorem 1.1 are open in the Whitney C^2 topology, in particular we shall assume, from now on, that Y is smooth. Let $\Pi : \mathbb{R}^3 \to \mathbb{R}$ be the orthogonal projection onto the first coordinate. By composing with a transformation T_θ if necessary (see Proposition 2.6) we may assume that $\Pi(\mathcal{A})$ is an unbounded interval. To simplify matters, let us suppose that $[b, \infty) \subset \Pi(\mathcal{A})$ and that $\Pi(A) \cap [b, \infty) = \emptyset$, where A is the compact face of $\mathcal{A}$.

By Thom's Transversality Theorem for jets [14], we may assume that $\mathcal{F}(f)$ has generic contact with the foliation $\mathcal{F}(\Pi)$. In this way, as f is a submersion,

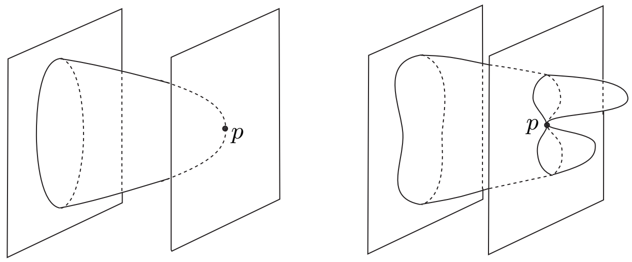
(a1) the contact manifold $T = \{(x, y, z) \in \mathbb{R}^3; f_y(x, y, z) = 0 = f_z(x, y, z)\}$ is a subset of $\{(x, y, z) \in \mathbb{R}^3 : f_x(x, y, z) \neq 0\}$ made up of regular curves;

(a2) there is a discrete subset Δ of T such that if $p \in T \setminus \Delta$, then Π, restricted to the leaf of $\mathcal{F}(f)$ passing through p, has a Morse-type singularity at p which is either a saddle point or an extremal (maximum or minimum) point. see Figure 2.

Then, if $a > b$ is large enough,

(b) for any $x \geq a$, the plane $\Pi^{-1}(x)$ intersects exactly one leave $L_x \subset \mathcal{A}$ of $\mathcal{F}(f)|_{\mathcal{A}}$ such that $\Pi(L_x) \cap (x, \infty) = \emptyset$. In other words, x is the supremum of the set $\Pi(L_x)$. Notice that L_x is a disc whose boundary is contained in the compact face of $\mathcal{A}$.

(c) if $x \geq a$ then $T_x = L_x \cap \Pi^{-1}(x)$ is contained in $T \cap \mathcal{A}$.



$p \in T \setminus \Delta$: center contact $\quad$ $p \in \Delta$: saddle contact

FIGURE 3.

(d) if $p \in T_x$ then $p \in T \setminus \Delta$ is a maximum point for the restriction $\Pi|_{L_x}$.

Notice that T_x is a finite set disjoint of Δ, for every $x \geq a$. Hence, the map $x \in [a, \infty) \longmapsto \#T_x$ is upper semi continuous, were $\#T_x$ denotes the cardinal number of T_x. To motivate what is claimed in (e) below, we observe that if, for some $x_0 \in [b, \infty)$ and for some $p \in T_{x_0}$, we had that $\#(T_{x_0}) > 1$ and $0 < f_x(p) < \min\{f_x(q) : q \in T_{x_0} \setminus \{p\}\}$, then, we would obtain that, for some $\epsilon > 0$ and for every $x \in (x_0 - \epsilon, x_0) \cup (x_0, x_0 + \epsilon)$, $\#T_x = 1$; in this way, there would exist a smooth curve $\eta : (x_0 - \varepsilon, x_0 + \varepsilon) \mapsto T$ such that $\eta(x_0) = p \in T_{x_0}$ and, for all $x \neq x_0$, $T_x = \{\eta(x)\}$.

Therefore, by $(b) - (d)$ and by using Thom's Transversality Theorem for jets, we may assume the following stronger statement:

(e) there is an increasing sequence $F = \{a_i\}_{i \geq 1}$ in $[a, +\infty)$, at most countable, such that if $x \in [a, +\infty) \setminus F$, then T_x is a one-point set.

If $x \in [a, +\infty) \setminus F$ and $T_x = \{(x, \eta_1(x), \eta_2(x))\}$, define $\eta : [a, +\infty) \setminus F \to T$ by $\eta(x) = (x, \eta_1(x), \eta_2(x))$. Observe that η is a smooth embedding and, since $f|_{\mathcal{A}}$ is continuous and bounded,

(f) $f \circ \eta$ extends continuously to a strictly monotone bounded map defined in $[a, +\infty)$ such that, for all $x \in [a, +\infty) \setminus F$, $f_x(\eta(x))$ has constant sign.

Therefore, there exists a real constant K such that

$$\begin{aligned} K &= \int_{a_1}^{+\infty} \frac{d}{dx}(f \circ \eta)(x) dx \\ &= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} \frac{d}{dx}(f \circ \eta)(x) dx \\ &= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} f_x(\eta(x)) dx. \end{aligned}$$

This and (f) imply that, for some sequence $x_n \to +\infty$,

$$\lim_{n \to +\infty} f_{x_n}(\eta(x_n)) = 0.$$

This contradiction, with the assumption that $\mathrm{Spec}(Y) \cap (-\varepsilon, \varepsilon) = \emptyset$, proves the theorem. $\square$

3. Proof of Theorems 1.2 and 1.3

To prove Theorem 1.2 we shall need the following.

Lemma 3.1 (Lemma 6.2.11 of [10])**.** *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring (A need not be commutative). Let $a \in A_d$, for some $d \geq 1$. Then $1 + a$ is invertible in A if and only if a is nilpotent.*

Proof of Theorem 1.2. We start as in the proof of Proposition 8.1.8 of [10]. By the Reduction Theorem (See [2], [9], [29]) it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F = (-X_1 + H_1, \ldots, -X_n + H_n)$$

where $X_i : \mathbb{C}^n \to \mathbb{C}$ denotes the canonical i-coordinate function, each H_i is either zero or homogeneous of degree 3 and JH (with $H = (H_1, H_2, \ldots, H_n)$) is nilpotent. Consider the polynomial map $\tilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$\tilde{F} = (ReF_1, ImF_1, \ldots, ReF_n, ImF_n).$$

So we have $\tilde{F} = -\tilde{X} + \tilde{H}$, where $\tilde{H}$ is homogeneous of degree 3. Since JH is nilpotent, $JF = -I + JH$ is invertible by Lemma 3.1 and $\det J\tilde{F} = |\det JF|^2 = 1$, whence $J\tilde{F}$ is invertible. So by Lemma 3.1, $J\tilde{H}$ is nilpotent and consequently $\mathrm{Spec}(\tilde{F}) = \{-1\}$.

Now we proceed as in the proof of Proposition 8.3 of [20]. By [21] and [22], we get that the set S_F has complex codimension 1, hence $S_{\tilde{F}}$ has real codimension 2. Now F is bijective if, and only if, $\tilde{F}$ is bijective, Therefore if the assumption of this theorem are satisfied, F will be bijective □

To prove Theorem 1.3 we shall need the following Jelonek results [20]:

Theorem 3.2. *if $Y : \mathbb{R}^n \to \mathbb{R}^n$ is a real polynomial mapping with nonzero Jacobian everywhere and* $\mathrm{codim}(S_Y) \geq 3$, *then Y is a bijection (and consequently $S_Y = \emptyset$).*

Theorem 3.3. *Let $Y : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial mapping. Then the set S_Y is closed, semi-algebraic and for every non-empty connected component $S \subset S_Y$ we have $1 \leq \dim(S) \leq n - 1$. Moreover, for every point $q \in S_Y$ there exists a polynomial mapping $\phi : \mathbb{R} \to S_Y$ such that $\phi(\mathbb{R})$ is a semi-algebraic curve containing $\{q\}$.*

The proof of the following lemma is easy and will be omitted.

Lemma 3.4. *Let $Y : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map such that* $\mathrm{Spec}(Y) \cap \{0\} = \emptyset$. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. If $Z = A \circ Y \circ A^{-1}$ then* $\mathrm{Spec}(Y) = \mathrm{Spec}(Z)$ *and $S_Z = A(S_Y)$.*

Proposition 3.5. *Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map such that* $\mathrm{Spec}(Y) \cap (-\varepsilon, \varepsilon) = \emptyset$, *for some $\varepsilon > 0$. If* $\mathrm{codim}(S_Y) \geq 2$ *then Y is a bijection.*

Proof. Suppose that Y is not bijective. By Theorem 3.2, we must have $\dim(S_Y) = 1$. Then by Theorem 3.3, we obtain that

(a) $Y(\mathbb{R}^3) \supset \mathbb{R}^3 \setminus S_Y$.

Therefore, using again Theorem 3.2, and Lemma 3.4, we way suppose that S_Y contains a regular curve meeting transversally the plane $\{x = a\}$ at the point $p = (a, b, c)$. In this way.

(b) the plane of $\{x = a\}$ contains a smooth embedded disc $D(a)$ such that $\{p\} = D(a) \cap S_Y$ and $C(a) \cap S_Y = \emptyset$, where $C(a)$ is the boundary of $D(a)$.

It is well known that there exists a positive integer K such that

(c) for all $q \in \mathbb{R}^3$, $\#Y^{-1}(q) \leq K$.

This implies that $Y^{-1}(C(a))$ is the union of finitely many embedded circles $C_1, C_2, \ldots, C_k$ contained in $f^{-1}(a)$. Each $Y_{|C_i} : C_i \to C(a)$ is a finite covering. As, by Theorem 1.1, each connected component of $f^{-1}(a)$ is a plane, we have that, for all $i = 1, 2, \ldots, k$, there exists a compact disc $D_i \subset f^{-1}(a)$ bounded by C_i. It follows that, for all $i = 1, 2, \ldots, k$, $Y(D_i) = D(a)$. As $D(a)$ is simply connected, for all $i \in \{1, 2, \ldots, k\}$, $Y_{|D_i} : D_i \to D(a)$ is a diffeomorphism. Hence, if $q \in C$, $\#Y^{-1}(q) = k$. As $D(a) \cap S_Y = \{p\}$ and $\#Y^{-1}$ is locally constant, $\#Y^{-1}$ must be identically equal to k in $D(a) \setminus \{p\}$ and therefore $Y^{-1}(D(a) - \{p\}) \subset \cup_{i=1}^k D_i$. As Y is a local diffeomorphism, by using a limiting procedure,

(d) for all $q \in D(a)$, $\#Y^{-1}(q) = k$ and so $Y^{-1}(D(a)) = \cup_{i=1}^k D_i$.

Notice that $D(a)$ can be taken of the form $D(a) = \{a\} \times D$, where D is a 2-disc of $\mathbb{R}^2$ centered at (b, c); in this way $C(a) = \{a\} \times \partial D$. We have that there exists $\varepsilon > 0$ small that.

(e) if $s \in [a - \varepsilon, a + \varepsilon]$, $D(s) = \{s\} \times D$ and $C(s) = \{s\} \times \partial D$, then $(s, b, c) = D(s) \cap S_Y$ and $C(s) \cap S_Y = \emptyset$.

Proceeding as above, we way find that for all $s \in [a - \varepsilon, a + \varepsilon]$ there are k embedded circles $C_1(s), C_2(s), \ldots, C_k(s)$, with $C_1(a) = C_1, C_2(a) = C_2, \ldots, C_k(a) = C_k$, contained in $f^{-1}(s)$ and such that $Y^{-1}(C(s)) = \cup_{i=1}^k C(k(s))$. Moreover each $C_i(s)$ depends continuously on s. Therefore,

(f) for all $s \in [a - \varepsilon, a + \varepsilon]$ and for all $i = 1, 2, \ldots, k$, there exists a compact disc $D_i(s) \subset f^{-1}(s)$ bounded by $C_i(s)$ such that $Y(D_i(s)) = D(s)$ and $D_i(s)$ depends continuously on s.

Proceeding as in the proof of (d) we obtain that

(g) for all $s \in [a - \varepsilon, a + \varepsilon]$ and for all $q \in D(s)$, $\#Y^{-1}(q) = k$ and $Y^{-1}(D(s)) = \cup_{i=1}^k D_i(s)$.

As $[a - \varepsilon, a + \varepsilon] \times D$ is a compact neighborhood of (a, b, c) and $Y^{-1}([a - \varepsilon, a + \varepsilon] \times D)$ is compact we obtain a contradiction with the assumption $p \in S_Y$. □

The proof of the following lemma can be found in [11] and [12]. We include it here for sake of completeness.

Lemma 3.6. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map such that* $\det(F'(x)) \neq 0$ *for all x in $\mathbb{R}^n$. Given $t \in \mathbb{R}$, let $F_t \colon \mathbb{R}^n \to \mathbb{R}^n$ denote the map $F_t(x) = F(x) - tx$. If there exists a sequence $\{t_m\}$ of real numbers converging to* 0 *such that every map $F_{t_m} \colon \mathbb{R}^n \to \mathbb{R}^n$ is injective, then F is injective.*

Proof. Choose $x_1, x_2 \in \mathbb{R}^n$ such that $F(x_1) = y = F(x_2)$. We will prove $x_1 = x_2$. By the Inverse Mapping Theorem, we may find neighborhoods U_1, U_2, V of x_1, x_2, y, respectively, such that, for $i = 1, 2$, $F|_{U_i} : U_i \to V$ is a diffeomorphism and $U_1 \cap U_2 = \emptyset$. If m is large enough, then $F_{t_m}(U_1) \cap F_{t_m}(U_2)$ will contain a neighborhood W of y. In this way, for all $w \in W$, $\#(F_{t_m}^{-1}(w)) \geq 2$. This contradiction with the assumptions, proves the lemma. □

Remark 3.7. Even if $n = 1$ and the maps F_{t_m} in Lemma 3.6 are smooth diffeomorphisms, we cannot conclude that F is a diffeomorphism. For instance, if $F : \mathbb{R} \to (0, 1)$ is an orientation reversing diffeomorphism, then for every $t > 0$, the map $F_t : \mathbb{R} \to \mathbb{R}$ (defined by $F_t(x) = F(x) - tx$) will be an orientation reversing global diffeomorphism.

Theorem 1.3. *Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map such that* $\mathrm{Spec}(Y) \cap [0, \varepsilon) = \emptyset$, *for some $\varepsilon > 0$. If* $\mathrm{codim}(S_Y) \geq 2$ *then Y is a bijection.*

Proof. We claim that for each $0 < t < \epsilon$, the map $Y_t \colon \mathbb{R}^3 \to \mathbb{R}^3$, given by $Y_t(x) = Y(x) - tx$, is injective.

In fact, as $D(Y_t)(x) = DY(x) - tI$, (where I is the Identity map), we obtain that if $0 < a < \min\{t, \epsilon - t\}$, then $\mathrm{Spec}(Y_t) \cap (-a, a) = \emptyset$. It follows immediately from Lemma 3.6 and Proposition 3.5 that Y is injective. The conclusion of this theorem is obtained by using Białynicki-Rosenlicht Theorem [3]. □

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